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ESTIMATES OF ELASTIC MODULI FOR GRANULAR MATERIAL WITH ANISOTROPIC RANDOM PACKING STRUCTURE

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Abstract—For randomly packed spheres, the packing structure can be characterized by a fabric tensor representing the orientation distribution of inter-particle contacts. The densities of inter-particle contacts are generally not the same for all inter-particle orientations, exhibiting the anisotropy of a packing structure. Utilizing the fabric tensor, elastic moduli of granular material with anisotropic packing structures are derived in explicit terms of inter-particle properties. The derivation is based on two different methods, namely the kinematic method and the static method. Results derived from kinematic and static methods provide, respectively, the upper and lower estimates of elastic moduli. Ranges of elastic moduli are discussed for granular materials with different inter-particle properties and fabric parameters. The close relationship between fabric tensor and material symmetry is demonstrated.

1. INTRODUCTION

In a micromechanical approach for constitutive modeling of granular material, the kinematic hypothesis of uniform strain has often been made to describe the displacement field of discrete particles (Digby, 1981; Walton, 1987; Bathurst and Rothenberg, 1988; Chang, 1988; Jenkins 1988). The kinematic hypothesis furnishes a useful relationship for estimating the movement of particles. However, it leads to an upper bound solution due to the kinematic constraint of the system. In order to remove the kinematic constraints, a number of efforts have been made to account for the effect of strain fluctuation in a granular material (Chang *et al.*, 1992; Chang, 1993; Misra and Chang, 1993).

However, contrary to the kinematic hypothesis, very little attention has been paid in the literature to the approach of employing a static hypothesis for granular systems. To this end, we propose a static hypothesis that delineates the distribution of forces at inter-particle contacts in a system of discrete particles. In this paper, following the approach of the static hypothesis, elastic moduli are derived in closed-form for assemblies of spheres with isotropic and anisotropic packing structures. The anisotropic packing structure is characterized by a fabric tensor representing the orientational distribution of inter-particle contacts. The closed-form expressions of elastic moduli are explicitly in terms of fabric parameters. Based on the closed-form expressions, the relationships between the fabric tensor and the material symmetry exhibited by the packing are discussed.

The derived moduli based on the static hypothesis correspond to a lower estimate solution. Compared with the results obtained from both static and kinematic hypotheses, we discuss the ranges of behavior for randomly packed granules with isotropic and anisotropic packing structures.

2. MICROSTRUCTURAL CONTINUUM MODEL

2.1. *Two particles in contact*

*Kinematics of two particles.*The granular system is envisioned to be composed of stiff particles. For practical purposes, the movement of particles can be approximately described by the kinematics of rigid particles. There are two modes of movement for a particle: translation, Δu_i^n, and rotation, $\Delta \omega_i^n$. The superscript 'n' refers to the nth particle. Based on the kinematics of two rigid particles of convex shape, the relative displacement $\Delta\delta_i^{nm}$ and the relative angular rotation $\Delta\theta_i^{nm}$ of particle 'm' to particle 'n' at the contact point are given by

$$\Delta\delta_i^{nm} = \Delta u_i^m - \Delta u_u^n + e_{ijk}(r_k^{mn}\Delta\omega_j^m - r_k^{nm}\Delta\omega_j^n) \tag{1}$$

$$\Delta\theta_i^{nm} = \Delta\omega_i^m - \Delta\omega_i^n, \tag{2}$$

where r_k^{nm} is the radius vector from the centroid of the nth particle to the point of contact with the mth particle. The quantity e_{ijk} is the permutation symbol used in tensor representation for the cross product of vectors.

Contact law. In a granular system, forces transmit through inter-particle contacts due to inter-particle displacement $\Delta\delta_i^{nm}$ and moments transmit through inter-particle contacts due to inter-particle angular rotation $\Delta\theta_i^{nm}$. For simplicity, we limit our discussion to convex-shape stiff particles with relatively small contact area. Thus we neglect the moment transmitting caused by the relative angular rotation at inter-particle contacts. Forces transmit through inter-particle contacts in both normal and tangential directions to contact surfaces. The resistance of an inter-particle contact to the relative displacement can be represented by two types of inter-particle stiffness: the normal inter-particle stiffness and the shear inter-particle stiffness. The shear inter-particle stiffness represents the resistance to sliding, while the normal inter-particle stiffness represents the resistance to compression of two particles.

For simplicity, we assume the inter-particle shear stiffnesses are the same in all directions on the contact plane. The inter-particle stiffnesses are denoted by k_n^{nm} and k_s^{nm} for the normal and the shear directions, respectively. The inter-particle flexibilities (i.e. the inverse of the inter-particle stiffnesses) are denoted by h_n^{nm} and h_s^{nm}, respectively.

The relative displacement, $\Delta\delta_i^{nm}$, and the contact force, Δf_i^{nm}, at the inter-particle contact can be expressed in a general relationship of incremental form as

$$\Delta f_i^{nm} = K_{ij}^{nm}\Delta\delta_j^{nm}; \quad \Delta\delta_i^{nm} = S_{ij}^{nm}\Delta f_j^{nm} \tag{3}$$

where the contact stiffness tensor K_{ij}^{nm} and the contact flexibility tensor S_{ij}^{nm} are given by

$$\begin{aligned} K_{ij}^{nm} &= k_n^{nm} n_i^{nm} n_j^{nm} + k_s^{nm}(s_i^{nm}s_j^{nm} + t_i^{nm}t_j^{nm}) \\ S_{ij}^{nm} &= h_n^{nm} n_i^{nm} n_j^{nm} + h_s^{nm}(s_i^{nm}s_j^{nm} + t_i^{nm}t_j^{nm}) \end{aligned} \tag{4}$$

in which **n**, **s**, and **t** are the basic unit vectors of the local coordinate system constructed at each contact, as shown in Fig. 1. The vector **n** is the outward normal to the contact plane. The other two orthogonal vectors **s** and **t** are on the contact plane.

2.2. *Stress and strain in a granular medium*

The relationship between the contact forces and the average stress of the assembly can be defined by employing the theorem of mean stress. To differentiate from the mean particle stress $\Delta\sigma_{ij}^n$, we define the stress $\Delta\tau_{ij}$ for an infinitesimal element within the particle, such that

$$\Delta\sigma_{ij}^n = \frac{1}{V^n}\int_{V^n}\Delta\tau_{ij}\,dv, \tag{5}$$

where V^n is the volume of the nth particle.

Using the equilibrium condition and divergence theorem, the volume integral in eqn (5) can be converted into a surface integral. For discrete forces on the boundary surface of

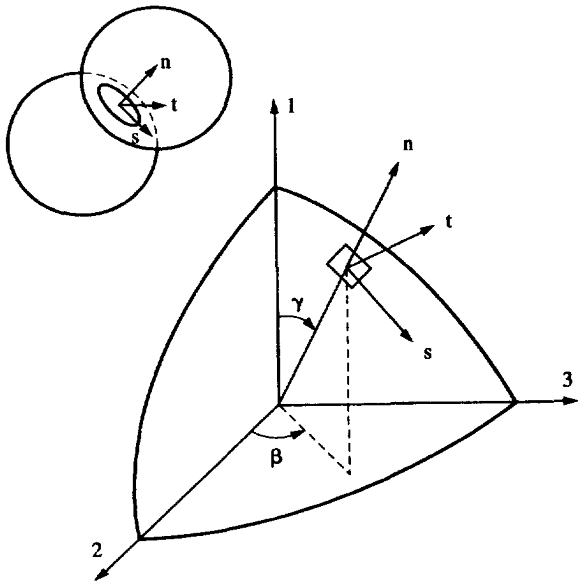

Fig. 1. Local coordinate system at an inter-particle contact.

the nth particle, the surface integral can be expressed by a summation over the boundary forces, given by (Chang and Liao, 1990)

$$\Delta\sigma_{ij}^{n} = \frac{1}{V^{n}} \sum_{m} r_i^{nm} \Delta f_j^{nm}. \tag{6}$$

The local stress $\Delta\sigma_{ij}^{n}$ for the nth particle is thus expressed as the volume average of the dyadic product of contact force Δf_j^{nm} and radius vector r_i^{nm} that connects the centroid of the nth particle to the contact point with the mth particle.

We further consider a given volume to be representative of the granular solid. The mean stress for the representative unit, denoted by $\Delta\sigma_{ij}$, is regarded as the volume average of the local stress at the particle level, such that

$$\Delta\sigma_{ij} = \frac{1}{V} \sum_{n} V^{n} \Delta\sigma_{ij}^{n} = \frac{1}{V} \sum_{n} \sum_{m} r_i^{nm} \Delta f_j^{nm}. \tag{7}$$

The mean stress of a representative volume can thus be written in terms of the inter-particle contact forces. Define a branch vector that connects the centroids of two particles, given by $L_i^{nm} = r_i^{nm} - r_i^{mn}$. An alternative form of eqn (7) can be obtained as follows:

$$\Delta\sigma_{ij} = \frac{1}{V} \sum_{c} L_i^{c} \Delta f_j^{c}. \tag{8}$$

This leads to the familiar results by Christoffersen *et al.* (1981) and Rothenberg and Selvadurai (1981) for the volume average of stress.

We now define the strain of aggregates, which is different from the strain of an infinitesimal element within a particle. The strain of aggregates is determined from the fields of translation and rotation for particles. For a small domain at the vicinity of the nth particle, the fields of translation and rotation for particles in this domain can be expressed by polynomial functions. The translation and rotation of the neighboring particle 'm' can be predicted from the information of the nth particle using Taylor's expansion

$$\Delta u_i^m = \Delta u_i^n + \Delta u_{i,j}^n L_j^{nm} + \tfrac{1}{2}\Delta u_{i,jk}^n L_j^{nm} L_k^{nm} + \dots$$

$$\Delta \omega_i^m = \Delta \omega_i^n + \Delta \omega_{i,j}^n L_j^{nm} + \tfrac{1}{2}\Delta \omega_{i,jk}^n L_j^{nm} L_k^{nm} + \dots, \tag{9}$$

where Δu_i^n and $\Delta \omega_i^n$ are, respectively, the translation and rotation of the nth particle. L_j^{nm} is the branch vector from the centroid of the nth particle to the centroid of the mth particle. $\Delta u_{i,j}^n$, $\Delta u_{i,jk}^n$, $\Delta \omega_{i,j}^n$, $\Delta \omega_{i,jk}^n$ are the gradients of the translation and rotation fields at the nth particle.

Using up to the first gradients and substituting eqn (9) into eqns (1) and (2), the inter-particle displacement and the inter-particle rotation can be expressed as

$$\Delta \delta_i^{nm} = (\Delta u_{i,j}^n - e_{ijk}\Delta \omega_k^n) L_j^{nm} + e_{ijk}\Delta \omega_{j,l}^n (r_k^{mn} L_l^{nm} - r_k^{nm} L_l^{nm}) \tag{10}$$

$$\Delta \theta_i^{nm} = \Delta \omega_{i,j}^n \mathrm{L}_j^{nm}. \tag{11}$$

The first gradient of rotation $\Delta \omega_{i,j}^n$ is termed the polar strain in micro-polar theory (Eringen, 1968 ; Chang and Ma, 1991). Since moments transmitting in the medium are not considered, we neglect the rotation gradient and eqn (11) vanishes. The granular medium is thus not treated as a micro-polar medium. With the absence of a rotation gradient, eqn (10) becomes

$$\Delta \delta_i^{nm} = (\Delta u_{i,j}^n - e_{ijk}\Delta \omega_k^n) L_j^{nm}. \tag{12}$$

Note that, in eqn (12), the particle rotation $\Delta \omega_i^n$ is still involved in the deformation mechanism, although the rotation gradient $\Delta \omega_{i,j}^n$ is neglected. During deformation of a granular material, the rotation of a particle consists of two components: the rigid body rotation $\Delta \Gamma_i^n$ and the particle spin $\Delta \psi_i^n$, i.e.

$$\Delta \omega_i^n = \Delta \psi_i^n + \Delta \Gamma_i^n. \tag{13}$$

The rigid body rotation is induced by the displacement field, given by

$$\Delta \Gamma_i^n = -\tfrac{1}{2} e_{ijk}\Delta u_{[j,k]}^n ; \quad \Delta u_{[i,j]}^n = -e_{ijk}\Delta \Gamma_k^n, \tag{14}$$

where $\Delta u_{[j,k]}$ is the skew-symmetric part of the displacement gradient, defined by

$$\Delta u_{[j,k]}^n = \tfrac{1}{2}(\Delta u_{j,k}^n - \Delta u_{k,j}^n). \tag{15}$$

If the effect of particle spin $\Delta \psi_i^n$ is not considered, particle rotation $\Delta \omega_i^n$ is solely induced from rigid body rotation. Thus, the medium is of the conventional type in solid mechanics. Since the effect of particle spin $\Delta \psi_i^n$ is considered in this paper, the granular medium is of a quasi-micro-polar type.

Observing the expression of eqn (12), we introduce an asymmetric strain, similar to the form used in the theory of a micro-polar medium, defined as (Chang and Liao, 1990; Chang and Ma, 1991):

$$\Delta \varepsilon_{ij}^n = \Delta u_{j,i}^n - e_{ijk}\Delta \omega_k^n \tag{16}$$

and the inter-particle displacement is related to the local asymmetric strain by

$$\Delta \delta_j^{nm} = \Delta \varepsilon_{ij}^n L_i^{nm}. \tag{17}$$

The symmetrical part of the strain $\Delta \varepsilon_{ij}^n$ is equal to the symmetrical part of the displacement gradient, representing the usual strain, i.e.

$$\Delta\varepsilon^n_{(ij)} = \tfrac{1}{2}(\Delta\varepsilon^n_{ij} + \Delta\varepsilon^n_{ji}) = \tfrac{1}{2}(\Delta u^n_{j,i} + \Delta u^n_{i,j}). \tag{18}$$

The skew-symmetric part of the strain $\Delta\varepsilon_{ij}$ is given by

$$\Delta\varepsilon^n_{[ij]} = \tfrac{1}{2}(\Delta\varepsilon^n_{ij} - \Delta\varepsilon^n_{ji}) = \tfrac{1}{2}(\Delta u^n_{j,i} - \Delta u^n_{i,j}) - e_{ijk}\Delta\omega^n_k. \tag{19}$$

Replacing the rigid body rotation [i.e. eqn (14)] into eqn (19), the skew-symmetric part of the strain becomes

$$\Delta\varepsilon^n_{[ij]} = e_{ijk}(\Delta\Gamma^n_k - \Delta\omega^n_k), \tag{20}$$

which represents the particle spin (i.e. the difference between the particle rotation and the rigid body rotation).

2.3. *Constitutive law derived from the kinematic hypothesis*

Kinematic hypothesis. Based on the kinematic assumption of uniform strain, we assume the local strain $\Delta\varepsilon^n_{ij}$ is equal to the overall strain of the representative volume $\Delta\varepsilon_{ij}$. Following eqn (17), a convenient kinematic relationship can be furnished that relates strain to the deformation $\Delta\delta^c_i$ at the contact, given by

$$\Delta\delta^c_j = \Delta\varepsilon_{ij}\mathbf{L}^c_i, \tag{21}$$

where $\mathbf{L}^c_i$ is the branch vector joining the centroids of the two contact particles.

Micromechanical description of stress. Due to a strain increment of a granular assembly, the work done from stress and strain of the packing is equal to the work done from forces and displacements at the inter-particle contacts, expressed as

$$\sigma_{ij}\Delta\varepsilon_{ij} = \frac{1}{V}\sum_c f^c_j \Delta\delta^c_j. \tag{22}$$

By substituting eqn (21) into eqn (22),

$$\Delta\varepsilon_{ij}\left(\sigma_{ij} - \frac{1}{V}\sum_c f^c_j L^c_i\right) = 0. \tag{23}$$

Since the incremental strain $\Delta\varepsilon_{ij}$ is arbitrarily chosen, in order to satisfy energy conservation, we have

$$\sigma_{ij} = \frac{1}{V}\sum_c f^c_j L^c_i. \tag{24}$$

Note that eqn (24) is in the same form as eqn (8).

Constitutive relationship. We seek for the overall stiffness tensor of a representative unit in the stress–strain relationship:

$$\Delta\sigma_{ij} = C_{ijkl}\Delta\varepsilon_{kl}. \tag{25}$$

The constitutive tensor can be derived from the following three relationships:

(1) the micromechanical description of stress [eqn (24)],
(2) the contact law [eqn (3)],

(3) the kinematic hypothesis [eqn (21)].

Using these three equations, the derived stiffness tensor C_{ijkl} is given by

$$C_{ijkl} = \frac{1}{V}\sum_{c} L_i^c K_{jl}^c L_k^c. \tag{26}$$

Note that the result is derived from a kinematic hypothesis on the deformation field. The deformation field is consistent with the applied strain and satisfies the compatibility condition for all particles within the representative volume. Therefore, the constitutive tensor corresponds to the Voigt upper bound.

2.4. *Constitutive law derived from static hypothesis*

Static hypothesis. Granular material is not homogeneous because it consists of two different phases, i.e. solid and pore space. Therefore, the traction on an inter-particle contact area cannot be obtained from the usual Cauchy's stress formula, given by

$$\Delta\tau_j^c = \Delta\sigma_{ij} n_i^c. \tag{27}$$

It is reasonable to expect that the distribution of forces on inter-particle contacts is influenced by the packing structure of the material. To this end, we introduce a static hypothesis that allows us to estimate the inter-particle contact forces directly from the mean stress of the packing, given by

$$\Delta f_j^c = \Delta\sigma_{ij} A_{ik} n_k^c, \tag{28}$$

where f_j^c is the inter-particle force and n_k^c is the unit normal vector outward from the contact surface on which the inter-particle force is acting. The tensor A_{ik} is a quantity related to packing structure, which will be determined subsequently.

The set of inter-particle forces, obtained from the static hypothesis, must be consistent with the micromechanical description of stress for the packing. Substituting eqn (28) into eqn (8), the following identity must be satisfied:

$$F_{ip} A_{kp} = \delta_{ik}, \tag{29}$$

where F_{ip} is a second-order tensor for the packing structure, given by

$$F_{ip} = \frac{1}{V}\sum_{c} L_i^c n_p^c. \tag{30}$$

Thus we denote the tensor A_{ip} as

$$A_{ip} = (F_{ip})^{-1}. \tag{31}$$

The packing structure tensor A_{ip} influences the distribution of inter-particle contact forces in the granular system.

Micromechanical description of strain. Similar to the stress tensor defined in eqn (24), the strain tensor in the granular assembly can be defined using the principle of energy conservation. In a granular assembly, the work done from stress and strain of the packing is equal to the work done from forces and displacements at the inter-particle contacts. For an increment of stress, the work can be expressed as

$$\Delta\sigma_{ij}\varepsilon_{ij} = \frac{1}{V}\sum_{c}\Delta f_j^c \delta_j^c. \tag{32}$$

By substituting eqn (28) into eqn (32),

$$\Delta\sigma_{ij}\left(\varepsilon_{ij} - \frac{1}{V}\sum_{c}\delta_j^c n_k^c A_{ik}\right) = 0. \tag{33}$$

Since the stress increment is arbitrary, it follows the expression of ε_{ij} in terms of contact displacement δ_i^c, given by

$$\varepsilon_{ij} = \frac{1}{V}\sum_{c}\delta_j^c n_k^c A_{ik}. \tag{34}$$

The derived strain in eqn (34), in terms of contact displacements, includes the second-order tensor A_{ik} which is a measure of packing structure.

Constitutive relationship. We now seek for the overall flexibility tensor of a representative unit in the stress–strain relationship:

$$\Delta\varepsilon_{ij} = S_{ijkl}\Delta\sigma_{kl}. \tag{35}$$

The constitutive tensor can be derived from the following three relationships:

(1) the micromechanical description of strain [eqn (34)],
(2) the contact law [eqn (3)],
(3) the static hypothesis [eqn (28)].

Using these three equations, the derived flexibility tensor S_{ijkl} is given by

$$S_{ijkl} = A_{im}A_{kn}\frac{1}{V}\sum_{c} n_m^c S_{jl}^c n_n^c. \tag{36}$$

Note that the result is derived from a static hypothesis on the field of contact forces. The contact force field is consistent with the applied stress. However, it is not guaranteed to satisfy the equilibrium condition for all particles within the representative volume. Therefore, the constitutive tensor is not necessarily a lower bound. It can only be regarded as a lower estimate.

3. GEOMETRICAL PROPERTIES OF PACKING STRUCTURE

Geometrical properties of granular packings have been of interest in various areas of engineering. Studies suggest that among the geometrical properties of granular packings, the important factors influencing the mechanical behavior are the radii of the particles, the contact number in the volume, the directional distribution of the branch vectors joining the centroids of two particles in contact, and the directional distribution of the inter-particle contacts (Gray, 1968; Oda *et al.* 1982; Shahinpoor, 1983). For packing of spherical particles, the orientations of branch vectors and contact vectors coincide. In the analysis of this paper, only packings of equal-sized spherical particles are considered.

For a suitably large representative volume with a large number of contacts, the summation of a quantity over all contacts can be expressed in an integral form. Let F be a quantity dependent upon the orientation of the contact; the summation of such a function over all contacts can be written as

$$\frac{1}{N}\sum_{c=1}^{N} F^c = \int_0^{2\pi}\int_0^{\pi} F(\gamma,\beta)\xi(\gamma,\beta)\sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta, \tag{37}$$

where N is the total number of inter-particle contacts, and the directional distribution density function $\xi(\gamma,\beta)$ satisfies

$$1 = \int_0^{2\pi}\int_0^{\pi} \xi(\gamma,\beta)\sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta. \tag{38}$$

Therefore, the stiffness tensor in eqn (26) based on the kinematic hypothesis can be written as follows:

$$C_{ijkl} = \frac{N}{V}\int_0^{2\pi}\int_0^{\pi} L_i(\gamma,\beta)K_{jl}(\gamma,\beta)L_k(\gamma,\beta)\xi(\gamma,\beta)\sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta. \tag{39}$$

For packing made of equal spheres, eqn (39) becomes

$$C_{ijkl} = \frac{4Nr^2}{V}\int_0^{2\pi}\int_0^{\pi} n_i(\gamma,\beta)K_{jl}(\gamma,\beta)n_k(\gamma,\beta)\xi(\gamma,\beta)\sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta. \tag{40}$$

Similarly, the flexibility tensor in eqn (36) based on the static hypothesis can be written as follows:

$$S_{ijkl} = A_{im}A_{kn}\frac{N}{V}\int_0^{2\pi}\int_0^{\pi} n_m(\gamma,\beta)S_{jl}(\gamma,\beta)n_n(\gamma,\beta)\xi(\gamma,\beta)\sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta, \tag{41}$$

where

$$F_{im} = (A_{im})^{-1} = \frac{2rN}{V}\int_0^{2\pi}\int_0^{\pi} n_i(\gamma,\beta)n_m(\gamma,\beta)\xi(\gamma,\beta)\sin\gamma\,\mathrm{d}\gamma\,\mathrm{d}\beta. \tag{42}$$

3.1. *Distribution density function of inter-particle contacts*

For modeling of random granular packings, it is convenient to represent the orientational distribution of inter-particle contacts, in three dimensions, as a spherical harmonics expansion (Chang and Misra, 1990b):

$$\xi(\gamma,\beta) = \frac{1}{4\pi}\left\{1+\sum_{k=2}^{\infty}{}'\left[a_{k0}P_k(\cos\gamma)+\sum_{m=1}^{k}P_k^m(\cos\gamma)(a_{km}\cos m\beta+b_{km}\sin m\beta)\right]\right\} \tag{43}$$

where γ and β are defined in Fig. 1. Here, Σ' = summation with respect to even indices only; $P_k(\cos\gamma)$ = kth Legendre polynomial; $P_k^m(\cos\gamma)$ = associated Legendre function; and a_{k0}, a_{km}, and b_{km} are fabric parameters. In order to ensure that the density function $\xi(\gamma,\beta)$ is centro-symmetric [i.e. $\xi(\gamma,\beta) = \xi(\pi-\gamma,\beta+\pi)$], only the even harmonics are admissible. It is evident that the first term, i.e. 1 in the expansion of eqn (43), represents a sphere, and the subsequent terms can be regarded as a function defined on the surface of the sphere. Further, since the Legendre polynomials and the associated Legendre functions are orthogonal to 1, it follows that eqn (43) is a density function

$$\int_{\Omega} \xi(\gamma, \beta)\, \mathrm{d}\Omega = 1. \tag{44}$$

In order to express the constitutive laws in a frame-indifferent form, it is useful to characterize the directional distributions of contact normals in tensorial form. The expression for the distribution density of contact normals, i.e. eqn (43), can be alternatively written as a Cartesian tensor equation:

$$\xi(n) = \frac{1}{4\pi}[1 + \mathbf{\Phi}_{ij}n_in_j + \mathbf{\Phi}_{ijkl}n_in_jn_kn_l + \ldots], \tag{45}$$

which represents a polynomial in terms of vector **n**, where the vector **n** is the unit normal vector at an inter-particle contact. In eqn (45), the tensor $\mathbf{\Phi}_{ij\ldots m}$ = a coefficient tensor of rank m of appropriate choice such that eqn (45) expands to eqn (43). It is evident that the coefficient tensor $\mathbf{\Phi}_{ij\ldots m}$ can be expressed in terms of the coefficients of eqn (43) (Kanatani, 1984). Thus, the coefficient tensor $\mathbf{\Phi}_{ij\ldots m}$ is a measure of the packing structure. The mth rank coefficient tensor $\mathbf{\Phi}_{ij\ldots m}$ is completely symmetric and traceless and can be obtained from experimentally measured data.

3.2. *Fabric tensor*

A second-rank symmetric tensor, termed as a fabric tensor, has been used by some investigators (Oda *et al.*, 1982; Satake, 1982; Cowin, 1985) as a representation of the packing structure. General forms of the fabric tensor have been discussed by Kanatani (1984).

For simplicity of subsequent derivation, a truncated form of the expansion in eqn (43), consisting of second-order terms, is used. The Legendre polynomial of degree two, i.e. $P_2(\cos\gamma)$, is given by

$$P_2(\cos\gamma) = \tfrac{1}{2}(3\cos^2\gamma - 1). \tag{46}$$

The associated Legendre function $P_2^2(\cos\gamma)$ can be obtained from the Rodrigues formula

$$P_k^m(x) = \frac{(1-x^2)^{m/2}}{2^k k!}\frac{d^{k+m}}{dx^{k+m}}(x^2-1)^k. \tag{47}$$

For $k = 2$ and $m = 2$, the above equation yields

$$P_2^2(\cos\gamma) = 3\sin^2\gamma. \tag{48}$$

Thus the truncated expansion is given as

$$\xi(\gamma, \beta) = \frac{1}{4\pi}[1 + \tfrac{1}{4}a_{20}(3\cos 2\gamma + 1) + 3\sin^2\gamma(a_{22}\cos 2\beta + b_{22}\sin 2\beta)]. \tag{49}$$

Alternatively, the above equation may be written as a Cartesian tensor equation

$$\xi(n) = \frac{1}{4\pi}[1 + \mathbf{\Phi}_{ij}n_in_j], \tag{50}$$

where $n = (\cos\gamma, \sin\gamma\cos\beta, \sin\gamma\sin\beta)$, and the coefficient tensor $\mathbf{\Phi}_{ij}$ is given by

$$[\mathbf{\Phi}_{ij}] = \begin{bmatrix} a_{20} & 0 & 0 \\ 0 & -\dfrac{a_{20}}{2}+3a_{22} & 3b_{22} \\ 0 & 3b_{22} & -\dfrac{a_{20}}{2}-3a_{22} \end{bmatrix}. \tag{51}$$

The first term in eqn (49), clearly represents the isotropic portion of the distribution, while the second term represents the anisotropic part. The fabric tensor is denoted to be $(\delta_{ij}+\mathbf{\Phi}_{ij})$ which combines the isotropic and the anisotropic parts to represent the distribution function of inter-particle contacts as follows:

$$\xi(n) = \frac{1}{4\pi}[(\delta_{ij}+\mathbf{\Phi}_{ij})n_i n_j]. \tag{52}$$

4. ELASTIC CONSTANTS FOR AN ANISOTROPIC PACKING STRUCTURE

The constitutive relationship for a random assembly can be obtained by substituting the density function given in eqn (50) into eqns (41) and (42).

4.1. *Based on kinematic hypothesis*

Substituting eqn (50) into eqn (41), the constitutive tensor becomes

$$C_{ijkl} = \frac{r^2 N}{\pi V}\int_0^{2\pi}\int_0^{\pi} K_{jl}(\delta_{rs}+\mathbf{\Phi}_{rs})n_i n_k n_r n_s \sin\gamma\, d\gamma\, d\beta. \tag{53}$$

Substituting the expression for the contact stiffness K_{jl} from eqn (4) into eqn (53) and carrying out the integration, a closed-form solution of the stiffness constants for the packing are derived. All inter-particle contact properties in the assembly are assumed to be independent of the stress state.

After integration, the stress–strain relationship, equivalent to eqn (25), can be expressed as follows:

$$\begin{Bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\sigma_{12} \\ \Delta\sigma_{21} \\ \Delta\sigma_{13} \\ \Delta\sigma_{31} \\ \Delta\sigma_{23} \\ \Delta\sigma_{32} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 & 0 & C_{1123} & C_{1132} \\ C_{2211} & C_{2222} & C_{2233} & 0 & 0 & 0 & 0 & C_{2223} & C_{2232} \\ C_{3311} & C_{3322} & C_{3333} & 0 & 0 & 0 & 0 & C_{3323} & C_{3332} \\ 0 & 0 & 0 & C_{1212} & C_{1221} & C_{1213} & C_{1231} & 0 & 0 \\ 0 & 0 & 0 & C_{2112} & C_{2121} & C_{2113} & C_{2131} & 0 & 0 \\ 0 & 0 & 0 & C_{1312} & C_{1321} & C_{1313} & C_{1331} & 0 & 0 \\ 0 & 0 & 0 & C_{3112} & C_{3121} & C_{3113} & C_{3131} & 0 & 0 \\ C_{2311} & C_{2322} & C_{2333} & 0 & 0 & 0 & 0 & C_{2323} & C_{2332} \\ C_{3211} & C_{3222} & C_{3233} & 0 & 0 & 0 & 0 & C_{3223} & C_{3232} \end{bmatrix} \begin{Bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{33} \\ \Delta\varepsilon_{12} \\ \Delta\varepsilon_{21} \\ \Delta\varepsilon_{13} \\ \Delta\varepsilon_{31} \\ \Delta\varepsilon_{23} \\ \Delta\varepsilon_{32} \end{Bmatrix}. \tag{54}$$

Table 1. Stiffness constants C_{ijkl}

$$C_{1111}=\frac{4r^2N}{15V}\left((3k_n+2k_s)+\frac{2a_{20}}{7}(6k_n+k_s)\right)$$

$$C_{2222}=\frac{4r^2N}{15V}\left((3k_n+2k_s)-\frac{a_{20}}{7}(6k_n+k_s)+\frac{6a_{22}}{7}(6k_n+k_s)\right)$$

$$C_{3333}=\frac{4r^2N}{15V}\left((3k_n+2k_s)-\frac{a_{20}}{7}(6k_n+k_s)-\frac{6a_{22}}{7}(6k_n+k_s)\right)$$

$$C_{1122}=\frac{4r^2N}{15V}\left((k_n-k_s)+\frac{a_{20}}{7}(k_n-k_s)+\frac{6a_{22}}{7}(k_n-k_s)\right)=C_{1221}$$

$$C_{1133}=\frac{4r^2N}{15V}\left((k_n-k_s)+\frac{a_{20}}{7}(k_n-k_s)-\frac{6a_{22}}{7}(k_n-k_s)\right)=C_{1331}$$

$$C_{2233}=\frac{4r^2N}{15V}\left((k_n-k_s)-\frac{2a_{20}}{7}(k_n-k_s)\right)=C_{2332}$$

$$C_{1212}=\frac{4r^2N}{15V}\left((k_n+4k_s)+\frac{a_{20}}{7}(k_n+13k_s)+\frac{6a_{22}}{7}(k_n-k_s)\right)$$

$$C_{2121}=\frac{4r^2N}{15V}\left((k_n+4k_s)+\frac{a_{20}}{7}(k_n-8k_s)+\frac{6a_{22}}{7}(k_n+6k_s)\right)$$

$$C_{1313}=\frac{4r^2N}{15V}\left((k_n+4k_s)+\frac{a_{20}}{7}(k_n+13k_s)-\frac{6a_{22}}{7}(k_n-k_s)\right)$$

$$C_{3131}=\frac{4r^2N}{15V}\left((k_n+4k_s)+\frac{a_{20}}{7}(k_n-8k_s)-\frac{6a_{22}}{7}(k_n+6k_s)\right)$$

$$C_{1213}=\frac{4r^2N}{15V}\left(\frac{6b_{22}}{7}(k_n-k_s)\right)=C_{1231}=C_{2113}=C_{1123}=C_{1132}$$

$$C_{2131}=\frac{4r^2N}{15V}\left(\frac{6b_{22}}{7}(k_n+6k_s)\right)$$

$$C_{2323}=\frac{4r^2N}{15V}\left((k_n+4k_s)-\frac{a_{20}}{7}(2k_n+5k_s)+6a_{22}k_s\right)$$

$$C_{3232}=\frac{4r^2N}{15V}\left((k_n+4k_s)-\frac{a_{20}}{7}(2k_n+5k_s)-6a_{22}k_s\right)$$

$$C_{2223}=\frac{4r^2N}{15V}\left(\frac{6b_{22}}{7}(3k_n-3k_s)\right)=C_{3332}$$

$$C_{2232}=\frac{4r^2N}{15V}\left(\frac{6b_{22}}{7}(3k_n+4k_s)\right)=C_{3323}$$

The stiffness constants C_{ijkl} are given in Table 1. It is noted that the stiffness constants C_{ijkl} are symmetric in matrix form. In Table 1, the number of contacts per unit volume of packing, N/V, can be expressed in terms of the coordination number $\hat{n}$ and void ratio e as follows:

$$\frac{N}{V}=\frac{3\hat{n}}{4\pi r^3(1+e)}. \tag{55}$$

Therefore the derived elastic moduli are functions of void ratio, coordination number, particle radius, inter-particle contact properties, and packing structure parameters.

4.2. *Based on static hypothesis*

Similarly, substituting eqn (50) into eqns (41) and (42), we obtain the flexibility tensor. By substituting eqn (50) into eqn (42), the second-order tensor A_{ij} is integrated and expressed in a matrix form as follows:

$$(A_{ij})^{-1} = F_{ij} = \frac{2rN}{15V}\begin{bmatrix} 5+2a_{20} & 0 & 0 \\ 0 & 5-a_{20}+6a_{22} & 6b_{22} \\ 0 & 6b_{22} & 5-a_{20}-6a_{22} \end{bmatrix}, \tag{56}$$

or, in a tensor form,

$$(A_{ij})^{-1} = F_{ij} = \frac{2rN}{3V}(\delta_{ij}+\tfrac{2}{5}\mathbf{\Phi}_{ij}). \tag{57}$$

Substituting eqn (50) into eqn (41), the flexibility tensor becomes

$$\mathbf{S}_{ijkl} = A_{im}A_{kn}\hat{\mathbf{S}}_{mjnl}, \tag{58}$$

where

$$\hat{\mathbf{S}}_{mjnl} = \frac{N}{4\pi V}\int_0^{2\pi}\int_0^{\pi} S_{jl}(\delta_{rs}+\mathbf{\Phi}_{rs})n_m n_n n_r n_s \sin\gamma\, d\gamma\, d\beta. \tag{59}$$

The fourth-order tensor $\hat{\mathbf{S}}_{mjnl}$ can be expressed in a matrix form as follows:

$$\hat{\mathbf{S}}_{mjnl} = \begin{bmatrix} \hat{S}_{1111} & \hat{S}_{1122} & \hat{S}_{1133} & 0 & 0 & 0 & 0 & \hat{S}_{1123} & \hat{S}_{1132} \\ \hat{S}_{2211} & \hat{S}_{2222} & \hat{S}_{2233} & 0 & 0 & 0 & 0 & \hat{S}_{2223} & \hat{S}_{2223} \\ \hat{S}_{3311} & \hat{S}_{3322} & \hat{S}_{3333} & 0 & 0 & 0 & 0 & \hat{S}_{3323} & \hat{S}_{3332} \\ 0 & 0 & 0 & \hat{S}_{1212} & \hat{S}_{1221} & \hat{S}_{1213} & \hat{S}_{1231} & 0 & 0 \\ 0 & 0 & 0 & \hat{S}_{2112} & \hat{S}_{2121} & \hat{S}_{2113} & \hat{S}_{2131} & 0 & 0 \\ 0 & 0 & 0 & \hat{S}_{1312} & \hat{S}_{1321} & \hat{S}_{1313} & \hat{S}_{1331} & 0 & 0 \\ 0 & 0 & 0 & \hat{S}_{3112} & \hat{S}_{3121} & \hat{S}_{3113} & \hat{S}_{3131} & 0 & 0 \\ \hat{S}_{2311} & \hat{S}_{2322} & \hat{S}_{2333} & 0 & 0 & 0 & 0 & \hat{S}_{2323} & \hat{S}_{2332} \\ \hat{S}_{3211} & \hat{S}_{3222} & \hat{S}_{3233} & 0 & 0 & 0 & 0 & \hat{S}_{3223} & \hat{S}_{3232} \end{bmatrix}. \tag{60}$$

The constants $\hat{S}_{mjnl}$ are given in Table 2. Note the similarity of eqn (59) and eqn (53). The value of $\hat{S}_{mjnl}$ can be obtained by dividing C_{mjnl} in Table 1 by $4r^2$ and substituting the contact stiffness constants with the contact flexibility constants.

5. CONSTITUTIVE EQUATION WITH CONVENTIONAL FORM

It is noted that the constitutive equations (25) and (35) are derived for general conditions of asymmetric stress and strain. The shear strain $\Delta\varepsilon_{ij}$ $(i \neq j)$ defined in this paper [see eqn (16)] is not the shear strain in the usual sense. To compare the derived moduli with those in conventional form, we define the following variables corresponding to the symmetric and skew-symmetric parts of the stress and strain, given by

$$\Delta\tau_{ij} = (\Delta\sigma_{ij}+\Delta\sigma_{ji})/2 \tag{61}$$

$$\Delta\sigma_{[ij]} = (\Delta\sigma_{ij}-\Delta\sigma_{ji})/2 \tag{62}$$

$$\Delta\gamma_{ij} = \Delta\varepsilon_{ij}+\Delta\varepsilon_{ji} \tag{63}$$

$$\Delta\psi_{ij} = \Delta\varepsilon_{ij}-\Delta\varepsilon_{ji}. \tag{64}$$

In terms of these variables, we reorganize the constitutive relationships of eqns (25) and (35). Since stress is always symmetric in the absence of polar stress, the skew-symmetric

Table 2. Constants $\hat{S}_{mjnl}$

$$\hat{S}_{1111} = \frac{N}{15V}\left((3h_n+2h_s)+\frac{2a_{20}}{7}(6h_n+h_s)\right)$$

$$\hat{S}_{2222} = \frac{N}{15V}\left((3h_n+2h_s)-\frac{a_{20}}{7}(6h_n+h_s)+\frac{6a_{22}}{7}(6h_n+h_s)\right)$$

$$\hat{S}_{3333} = \frac{N}{15V}\left((3h_n+2h_s)-\frac{a_{20}}{7}(6h_n+h_s)-\frac{6a_{22}}{7}(6h_n+h_s)\right)$$

$$\hat{S}_{1122} = \frac{N}{15V}\left((h_n-h_s)+\frac{a_{20}}{7}(h_n-h_s)+\frac{6a_{22}}{7}(h_n-h_s)\right) = \hat{S}_{1221}$$

$$\hat{S}_{1133} = \frac{N}{15V}\left((h_n-h_s)+\frac{a_{20}}{7}(h_n-h_s)-\frac{6a_{22}}{7}(h_n-h_s)\right) = \hat{S}_{1331}$$

$$\hat{S}_{2233} = \frac{N}{15V}\left((h_n-h_s)-\frac{2a_{20}}{7}(h_n-h_s)\right) = \hat{S}_{2332}$$

$$\hat{S}_{1212} = \frac{N}{15V}\left((h_n+4h_s)+\frac{a_{20}}{7}(h_n+13h_s)+\frac{6a_{22}}{7}(h_n-h_s)\right)$$

$$\hat{S}_{2121} = \frac{N}{15V}\left((h_n+4h_s)+\frac{a_{20}}{7}(h_n-8h_s)+\frac{6a_{22}}{7}(h_n+6h_s)\right)$$

$$\hat{S}_{1313} = \frac{N}{15V}\left((h_n+4h_s)+\frac{a_{20}}{7}(h_n+13h_s)-\frac{6a_{22}}{7}(h_n-h_s)\right)$$

$$\hat{S}_{3131} = \frac{N}{15V}\left((h_n+4h_s)+\frac{a_{20}}{7}(h_n-8h_s)-\frac{6a_{22}}{7}(h_n+6h_s)\right)$$

$$\hat{S}_{1213} = \frac{N}{15V}\left(\frac{6b_{22}}{7}(h_n-h_s)\right) = \hat{S}_{1231} = \hat{S}_{2113} = \hat{S}_{1123} = \hat{S}_{1132}$$

$$\hat{S}_{2131} = \frac{N}{15V}\left(\frac{6b_{22}}{7}(h_n+6h_s)\right)$$

$$\hat{S}_{2323} = \frac{N}{15V}\left((h_n+4h_s)-\frac{a_{20}}{7}(2h_n+5h_s)+6a_{22}h_s\right)$$

$$\hat{S}_{3232} = \frac{N}{15V}\left((h_n+4h_s)-\frac{a_{20}}{7}(2h_n+5h_s)-6a_{22}h_s\right)$$

$$\hat{S}_{2223} = \frac{N}{15V}\left(\frac{6b_{22}}{7}(3h_n-3h_s)\right) = \hat{S}_{3332}$$

$$\hat{S}_{2232} = \frac{N}{15V}\left(\frac{6b_{22}}{7}(3h_n+4h_s)\right) = \hat{S}_{3323}$$

part of stress $\Delta\sigma_{[ij]} = 0$. On this basis, we can decompose the nine constitutive equations in the form of a 9×9 constitutive matrix into two sets of equations: (1) six constitutive equations in the conventional form of a 6×6 constitutive matrix, and (2) three equations of constraint for the skew-symmetric strain. The decomposition is carried out for both hypotheses with the condition of fabric parameter $b_{22} = 0$.

5.1. *Based on kinematic hypothesis*

The constitutive equation in conventional form is given by

$$\begin{Bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\tau_{12} \\ \Delta\tau_{31} \\ \Delta\tau_{23} \end{Bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{2211} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{3311} & C_{3322} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_{1212} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{C}_{3131} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{C}_{2323} \end{bmatrix} \begin{Bmatrix} \Delta\varepsilon_{11} \\ \Delta\varepsilon_{22} \\ \Delta\varepsilon_{33} \\ \Delta\gamma_{12} \\ \Delta\gamma_{31} \\ \Delta\gamma_{23} \end{Bmatrix}, \quad (65)$$

where

$$\bar{C}_{1212} = \frac{1}{4}\left(C_{1212}+C_{2121}+2C_{1221}-\frac{4r^2N}{15V}\frac{(3a_{20}-6a_{22})^2}{(10+a_{20}+6a_{22})}k_s\right)$$

$$\bar{C}_{3131} = \frac{1}{4}\left(C_{3131}+C_{1313}+2C_{3113}-\frac{4r^2N}{15V}\frac{(-3a_{20}-6a_{22})^2}{(10+a_{20}-6a_{22})}k_s\right)$$

$$\bar{C}_{2323} = \frac{1}{4}\left(C_{2323}+C_{3232}+2C_{2332}-\frac{4r^2N}{15V}\frac{(12a_{22})^2}{(10-2a_{20})}k_s\right) \quad (66)$$

in which C_{ijkl} can be obtained from Table 1.

The three equations of constraint for the skew-symmetric strain are given by

$$\begin{Bmatrix}\Delta\psi_{12}\\ \Delta\psi_{31}\\ \Delta\psi_{23}\end{Bmatrix} = \begin{bmatrix}\bar{D}_{1212} & 0 & 0\\ 0 & \bar{D}_{3131} & 0\\ 0 & 0 & \bar{D}_{2323}\end{bmatrix}\begin{Bmatrix}\Delta\gamma_{12}\\ \Delta\gamma_{31}\\ \Delta\gamma_{23}\end{Bmatrix}, \quad (67)$$

where

$$\bar{D}_{1212} = -\frac{k_s^3}{8}(10-2a_{20})(10+a_{20}-6a_{22})(3a_{20}-6a_{22})$$

$$\bar{D}_{3131} = -\frac{k_s^3}{8}(10-2a_{20}-6a_{22})(10+a_{20}+6a_{22})(12a_{22})$$

$$\bar{D}_{2323} = -\frac{k_s^3}{8}(10-2a_{20})(10+a_{20}+6a_{22})(-3a_{20}-6a_{22}). \quad (68)$$

For materials with isotropic packing structure (i.e. a_{20}, $a_{22} = 0$), the constants $\bar{D}_{ijkl}$ in eqn (68) become zero, indicating zero skew-symmetric strain. However, for materials with anisotropic structure, the skew-symmetric strain is a function of the symmetric strain. Thus, net spin of particles occurs. Note that the derived elastic shear moduli in eqn (66), due to the consideration of particle spin in the present formulation, are smaller than those derived previously by Chang and Misra (1990b).

5.2. *Based on the static hypothesis*

Under the condition $b_{22} = 0$, the second-order tensor $\mathbf{A}_{im}$ in eqn (56) is reduced to a diagonal matrix. We then have

$$\begin{Bmatrix}\Delta\varepsilon_{11}\\ \Delta\varepsilon_{22}\\ \Delta\varepsilon_{33}\\ \Delta\gamma_{12}\\ \Delta\gamma_{31}\\ \Delta\gamma_{23}\end{Bmatrix} = \begin{bmatrix}A_{11}A_{11}\hat{S}_{1111} & A_{11}A_{22}\hat{S}_{1122} & A_{11}A_{33}\hat{S}_{1133} & 0 & 0 & 0\\ A_{22}A_{11}\hat{S}_{2211} & A_{22}A_{22}\hat{S}_{2222} & A_{22}A_{33}\hat{S}_{2233} & 0 & 0 & 0\\ A_{33}A_{11}\hat{S}_{3311} & A_{33}A_{22}\hat{S}_{3322} & A_{33}A_{33}\hat{S}_{3333} & 0 & 0 & 0\\ 0 & 0 & 0 & \bar{S}_{1212} & 0 & 0\\ 0 & 0 & 0 & 0 & \bar{S}_{3131} & 0\\ 0 & 0 & 0 & 0 & 0 & \bar{S}_{2323}\end{bmatrix}\begin{Bmatrix}\Delta\sigma_{11}\\ \Delta\sigma_{22}\\ \Delta\sigma_{33}\\ \Delta\tau_{12}\\ \Delta\tau_{31}\\ \Delta\tau_{23}\end{Bmatrix}, \quad (69)$$

where

$$\bar{S}_{1212} = A_{11}A_{11}\hat{S}_{1212}+A_{22}A_{22}\hat{S}_{2121}+2A_{11}A_{22}\hat{S}_{1221}$$
$$S_{3131} = A_{33}A_{33}\hat{S}_{3131}+A_{11}A_{11}\hat{S}_{1313}+2A_{33}A_{11}\hat{S}_{3113}$$
$$\bar{S}_{2323} = A_{22}A_{22}\hat{S}_{2323}+A_{33}A_{33}\hat{S}_{3232}+2A_{22}A_{33}\hat{S}_{2332} \tag{70}$$

in which $\hat{S}_{ijkl}$ can be found in Table 2.

The three equations of constraint for the skew-symmetric strain are given by

$$\begin{Bmatrix} \Delta\psi_{12} \\ \Delta\psi_{31} \\ \Delta\psi_{23} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \bar{T}_{1212} & \bar{T}_{1231} & 0 \\ 0 & 0 & 0 & \bar{T}_{3112} & \bar{T}_{3131} & 0 \\ \bar{T}_{2311} & \bar{T}_{2322} & \bar{T}_{2333} & 0 & 0 & \bar{T}_{2323} \end{bmatrix} \begin{Bmatrix} \Delta\sigma_{11} \\ \Delta\sigma_{22} \\ \Delta\sigma_{33} \\ \Delta\tau_{12} \\ \Delta\tau_{31} \\ \Delta\tau_{23} \end{Bmatrix}, \tag{71}$$

where

$$\bar{T}_{1212} = A_{11}A_{11}\hat{S}_{1212}-A_{22}A_{22}\hat{S}_{2121}$$
$$\bar{T}_{1231} = A_{11}A_{11}\hat{S}_{1213}+A_{11}A_{33}\hat{S}_{1231}-A_{11}A_{22}\hat{S}_{2113}-A_{22}A_{33}\hat{S}_{2131}$$
$$\bar{T}_{3112} = A_{33}A_{11}\hat{S}_{3112}+A_{33}A_{22}\hat{S}_{3121}-A_{11}A_{11}\hat{S}_{1312}-A_{11}A_{22}\hat{S}_{1321}$$
$$\bar{T}_{3131} = A_{33}A_{33}\hat{S}_{3131}-A_{11}A_{11}\hat{S}_{1313}$$
$$\bar{T}_{2311} = A_{22}A_{11}\hat{S}_{2311}-A_{33}A_{11}\hat{S}_{3211}$$
$$\bar{T}_{2322} = A_{22}A_{22}\hat{S}_{2322}-A_{33}A_{22}\hat{S}_{3222}$$
$$\bar{T}_{2333} = A_{22}A_{33}\hat{S}_{2333}-A_{33}A_{33}\hat{S}_{3233}$$
$$\bar{T}_{2323} = A_{22}A_{22}\hat{S}_{2323}-A_{33}A_{33}\hat{S}_{3232}. \tag{72}$$

Equation (72) shows that the constants $\bar{T}_{ijkl}$ are zero for materials with isotropic structure. Therefore particle spin (i.e. the skew-symmetric strain) does not occur. However, for materials with anisotropic structure, the skew-symmetric strain is no longer zero. The magnitude of particle spin is a function of the applied stress.

6. MATERIAL SYMMETRY

The fabric parameters a_{20}, a_{22}, and b_{22} represent types of packing structure. Using the derived expressions for elastic moduli and appropriate values of the fabric parameters a_{20}, a_{22}, and b_{22}, the stiffness and flexibility tensors for packings with various types of material symmetries can be represented.

6.1. *Isotropy*

If all the fabric parameters are chosen to be zeros (i.e. $a_{20} = a_{22} = b_{22} = 0$), the fabric tensor reduces to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{73}$$

which represents equal densities of contacts in all orientations. Under this condition, the derived constitutive constants in Tables 1 and 2 represent an isotropic material symmetry as follows:

$$\begin{bmatrix} \ast & \ast & \ast & 0 & 0 & 0 \\ . & \ast & \ast & 0 & 0 & 0 \\ . & . & \ast & 0 & 0 & 0 \\ . & . & . & \blacklozenge & 0 & 0 \\ . & . & . & . & \blacklozenge & 0 \\ . & . & . & . & . & \blacklozenge \end{bmatrix}, \tag{74}$$

where $\ast$ = non-zero components, $\ast - \ast$ = equal components, $\blacklozenge = \frac{1}{2}(C_{1111} - C_{1122})$ based on the kinematic hypothesis, and $\blacklozenge = 2(S_{1111} - S_{1122})$ based on the static hypothesis.

The Young's moduli, Poisson's ratio and shear moduli, based on kinematic and static hypotheses, are expressed in terms of contact stiffness as follows.

Based on the kinematic hypothesis

$$E = \frac{2k_n}{rv}\left(\frac{2+3\alpha}{4+\alpha}\right); \quad G = \frac{k_n}{5rv}(2+3\alpha);$$

$$B = \frac{2k_n}{3rv}; \quad \nu = \frac{1-\alpha}{4+\alpha} \tag{75}$$

where $\alpha = k_s/k_n$, $v = 3V/2Nr^3$, N is the total number of inter-particle contacts in the representative volume V, and r is the radius of particles.

For the possible range of stiffness ratios α from 0 to ∞, the Poisson's ratio ranges from 1/4 to -1. The range of Young's modulus is $(1 \sim 6)\ k_n/rv$. The range of shear modulus is $(2/5 \sim \infty)\ k_n/rv$. The Bulk modulus is independent of k_s. These results are the same as those obtained by Chang and Ma (1992).

Based on the static hypothesis

$$E = \frac{10k_n}{rv}\left(\frac{\alpha}{2+3\alpha}\right); \quad G = \frac{5k_n}{rv}\left(\frac{\alpha}{3+2\alpha}\right);$$

$$B = \frac{2k_n}{3rv}; \quad \nu = \frac{1-\alpha}{2+3\alpha}. \tag{76}$$

For the possible range of stiffness ratios α from 0 to ∞, the Poisson's ratio ranges from 1/2 to $-1/3$. The range of Young's modulus is $(0 \sim 10/3)k_n/rv$. The range of shear modulus is $(0 \sim 5/2)k_n/rv$. Note that the expressions for the bulk modulus derived from both kinematic and static hypotheses are identical.

From the derived results, it is evident that elastic moduli are functions of geometrical factors (i.e. the radii of the particles, the contact number in the volume, and the fabric parameters) and the contact stiffness (i.e. k_n and k_s). To illustrate the effect of the ratio k_s/k_n on the stiffness properties, Young's modulus and Poisson's ratio *versus* the ratio k_s/k_n for materials with isotropic packing structure are shown in Figs 2 and 3, respectively.

Observed from Fig. 2, an assembly with rough particle surface has a higher Young's modulus than that with smooth particle surface. For assembly with perfectly smooth particle surface (i.e. $k_s/k_n = 0$), Young's modulus is zero and Poisson's ratio is 0.5, predicted from static hypothesis. The granular material does not have any load-carrying capacity, representing a fluid-like behavior. On the other hand, the results predicted from the kinematic hypothesis, for assemblies with a perfectly smooth particle surface, still remain moderate in stiffness. For assemblies with k_s/k_n approaching 1 (i.e. rough particle surface), the Poisson's ratio approaches zero. In cases of k_s/k_n greater than 1 (i.e. a very rough particle surface), the Poisson's ratio can be negative.

Given the same stiffness ratio, Fig. 2 indicates that moduli based on the static hypothesis are smaller than those based on the kinematic hypothesis. It is noted that the upper

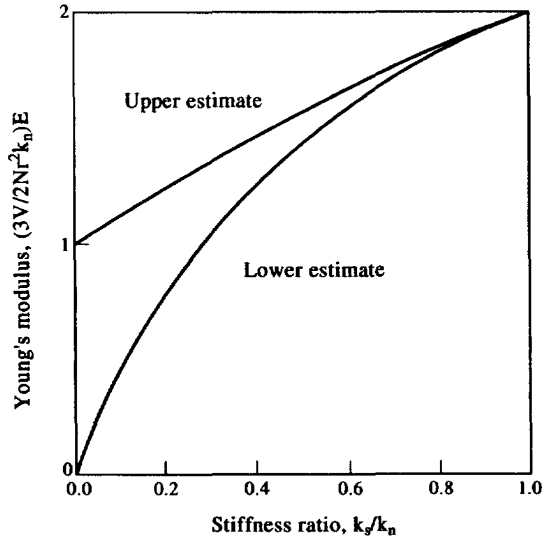

Fig. 2. Predicted Young's modulus with various stiffness ratios.

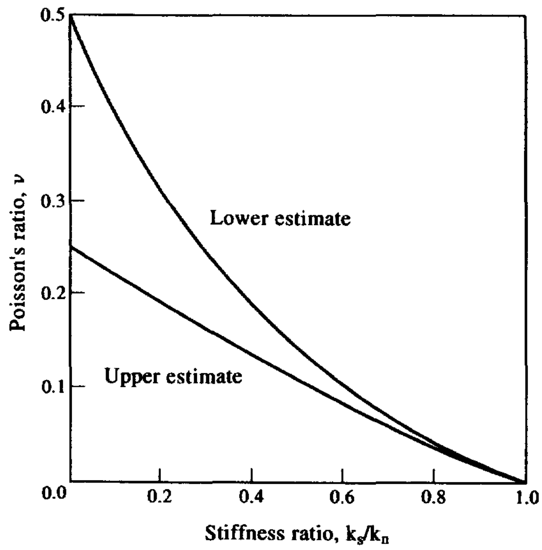

Fig. 3. Predicted Poisson's ratio with various stiffness ratios.

estimate is a true upper bound, while the lower estimate is not a true lower bound as discussed previously. Therefore, the ranges in Figs 2 and 3 do not fully cover the mechanical behavior for materials with isotropic packing structures. However, the two estimates provide an approximate range of the true behavior. For materials with isotropic packing structures, the range of behavior for materials with smooth particles is substantially larger than that for materials with rough particles. This is caused by the extra degree of freedom provided by the smooth particle surface. In reality, detailed information about the packing structures is not available. Therefore, the estimates of an approximate range are useful for understanding the behavior of random media.

6.2. *Transverse isotropy*

If the fabric parameters are chosen such that $a_{22} = b_{22} = 0$, the fabric tensor reduces to

$$\begin{bmatrix} 1+a_{20} & 0 & 0 \\ 0 & 1-\dfrac{a_{20}}{2} & 0 \\ 0 & 0 & 1-\dfrac{a_{20}}{2} \end{bmatrix}. \quad (77)$$

The fabric tensor in this form represents a fabric structure of transverse isotropy. The contact distribution in direction 2 is the same as that in direction 3, but different from that in direction 1 in the coordinate system shown in Fig. 1. For a positive value of a_{20}, there are more contacts oriented in direction 1. Under this condition, the derived constitutive constant in Tables 1 and 2 represent a transverse isotropic material symmetry as follows:

$$\begin{bmatrix} \ast & \ast & \ast & 0 & 0 & 0 \\ . & \ast & \ast & 0 & 0 & 0 \\ . & . & \ast & 0 & 0 & 0 \\ . & . & . & \ast & 0 & 0 \\ . & . & . & . & \ast & 0 \\ . & . & . & . & . & \blacklozenge \end{bmatrix}, \quad (78)$$

where $\ast$ = non-zero components, $\ast$—$\ast$ = equal components, $\blacklozenge = \frac{1}{2}(C_{1111}-C_{1122})$ based on the kinematic hypothesis, and $\blacklozenge = 2(S_{1111}-S_{1122})$ based on the static hypothesis.

To investigate the effect of degree of fabric anisotropy on the mechanical properties, the shear moduli in the plane of isotropy (G_{1212}) and in the plane perpendicular to it (G_{2323}) for transverse isotropic packings are plotted against the parameter a_{20} in Fig. 4. The ratio of k_s/k_n is chosen to be 1/2 for this case.

Given the same number of contacts per volume, when the fabric parameter a_{20} increases, the contact orientation increases in direction 1 while it decreases in directions 2 and 3. This leads to an increase in G_{1212} and a decrease in G_{2323}. The range bounded by upper and lower estimates is not affected by the fabric parameter a_{20}.

6.3. *Orthotropy*

Granular packings with orthotropic material symmetry can be represented by choosing the fabric parameter a_{20} and a_{22} to be nonzero as follows:

$$\begin{bmatrix} 1+a_{20} & 0 & 0 \\ 0 & 1-\dfrac{a_{20}}{2}+3a_{22} & 0 \\ 0 & 0 & 1-\dfrac{a_{20}}{2}-3a_{22} \end{bmatrix}. \quad (79)$$

The fabric tensor in this form represents a fabric structure of orthotropy. The contact distributions are different in the three axis directions. Under this condition, the derived constitutive constants in Tables 1 and 2 represent an orthotropic material symmetry as follows:

$$\begin{bmatrix} \ast & \ast & \ast & 0 & 0 & 0 \\ . & \ast & \ast & 0 & 0 & 0 \\ . & . & \ast & 0 & 0 & 0 \\ . & . & . & \ast & 0 & 0 \\ . & . & . & . & \ast & 0 \\ . & . & . & . & . & \ast \end{bmatrix}. \quad (80)$$

Clearly, the degree of anisotropy in the packing structure has a significant effect on the anisotropy in the mechanical behavior.

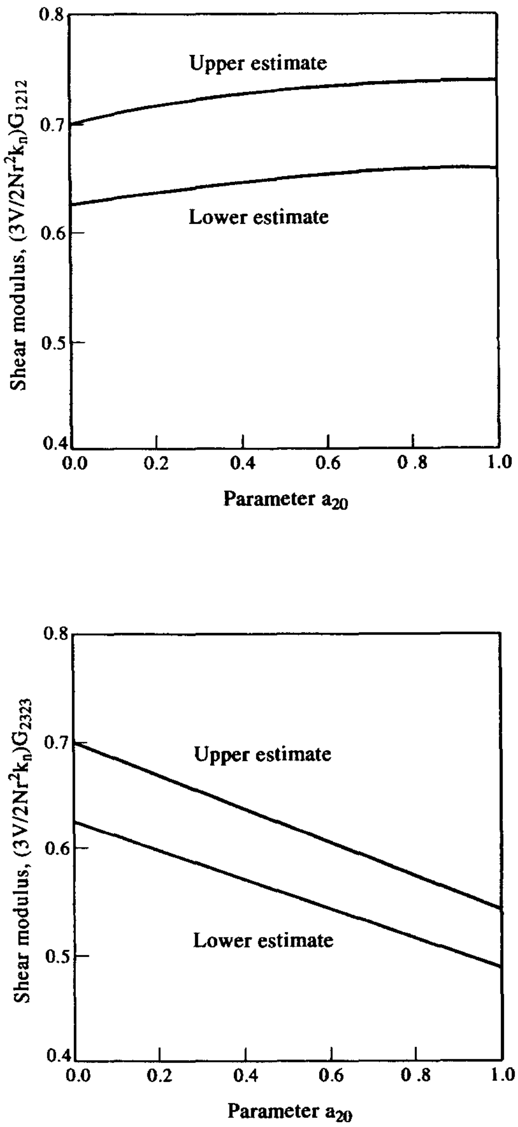

Fig. 4. Effect of fabric parameter a_{20} on the predicted shear moduli.

7. CONCLUDING REMARKS

In this paper, we derive the elastic moduli for random packings with anisotropic structure based on the proposed static hypothesis. The anisotropic structure is characterized by a fabric tensor representing the distribution density function of inter-particle contact. The closed-form expression for moduli shows a direct relationship between the fabric tensor and the material symmetry exhibited by the packing, such as isotropy, transverse isotropy, and orthotropy.

Elastic moduli for assemblies are significantly affected by the inter-particle properties. For example, the Poisson's ratio is close to 0.5 for assemblies with smooth particle surfaces. It approaches 0 for assemblies with rough particle surfaces, and becomes negative for assemblies with very rough particle surfaces.

The derived results based on the kinematic hypothesis is an upper bound solution, while the results based on the static hypothesis is a lower estimate as discussed previously. Results from the two methods provide an approximate range of the true behavior. The proposed method based on the static hypothesis is not only an interesting theoretical counterpart of the method based on the kinematic hypothesis, but also a useful tool for estimating ranges of behavior for randomly packed assemblies.

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REFERENCES

Bathurst, R. J. and Rothenberg, L. (1988). Micromechanical aspects of isotropic granular assemblies with linear contact interactions. *ASME J. Appl. Mech.* **55**, 17–23.

Chang, C. S. (1988). Micromechanical modelling of constitutive equation for granular material. In *Micromechanics of Granular Materials* (Edited by J. T. Jenkins and M. Satake), pp. 271–278. Elsevier Science Publishers.

Chang, C. S. (1993). Micromechanics modeling for deformation and failure of granulates with frictional contacts. In *Mechanics of Materials*, Vol. 16, No. 1–2, pp. 13–24. Elsevier Science Publishers, Amsterdam.

Chang, C. S. and Liao, C. (1990). Constitutive relations for particulate medium with the effect of particle rotation. *Int. J. Solids Structures* **26**(4), 437–453.

Chang, C. S. and Ma, L. (1991). A micromechanical-base micro-polar theory for deformation of granular solids. *Int. J. Solids Structures* **28**(1), 67–86.

Chang, C. S. and Ma, L. (1992). Elastic material constants for isotropic granular solids with particle rotation. *Int. J. Solids Structures* **29**(8), 1001–1018.

Chang, C. S. and Misra, A. (1990a). Application of uniform strain theory to heterogeneous granular solids. *ASCE J. Engng Mech. Div.* **116**(10), 2310–2328.

Chang, C. S. and Misra, A. (1990b). Packing structure and mechanical properties of granulates. *ASCE J. Engng Mech. Div.* **116**(5), 1077–1093.

Chang, C. S., Misra, A. and Acheampon, K. (1992). Elastoplastic deformation of granulates with frictional contacts. *ASCE J. Engng Mech.* **118**(8), 1692–1708.

Christoffersen, J., Mehrabadi, M. M. and Nemat-Nasser, S. (1981). A micromechanical description of granular material behavior. *J. Appl. Mech.* **48**(2), 339–344.

Cowin, S. C. (1985). The relationship between the elastic tensor and the fabric tensor. *Mech. Mater.* **4**(2), 137–147.

Digby, P. J. (1981). The effective elastic moduli of porous granular rock. *ASME J. Appl. Mech.* **48**(4), 803–808.

Eringen, A. E. (1968). Theory of micropolar elasticity. In *Fracture—An Advanced Treatise* (Edited by H. Liebowitz), Vol. II, Chapter 7, pp. 621–693. Academic Press, New York.

Gray, W. A. (1968). *The Packing of Solid Particles*. Chapman and Hall, London.

Jenkins, J. T. (1988). Volume change in small strain axisymmetric deformations of a granular material. *Micromechanics of Granular Materials* (Edited by M. Satake and J. T. Jenkins), pp. 143–152. Elsevier, Amsterdam, The Netherlands.

Kanatani, K. (1984). Distribution of directional data and fabric tensors. *Int. J. Engng Sci.* **22**(2), 149–164.

Misra, A. and Chang, C. S. (1993). Effective elastic moduli of heterogeneous granular solids. *Int. J. Solids Structures* **30**(18), 2547–2566.

Oda, M., Nemat-Nasser, S. and Meharabadi, M. M. (1982). A statistical study of fabric in a random assembly of spherical granules. *Int. J. Num. Anal. Meth. Geomech.* **6**(1), 77–94.

Rothenberg L. and Selvadurai, A. P. S. (1981). Micromechanical definition of the Cauchy stress tensor for particulate media. In *Mechanics of Structured Media* (Edited by A. P. S. Selvadurai), pp. 469–486. Elsevier, Amsterdam, The Netherlands.

Satake, M. (1982). Fabric tensor in granular materials. In *Proc. IUTAM Symp. on Deformation and Failure of Granular materials* (Edited by P. A. Vermeer and H. J. Luger), pp. 63–68. A. A. Balkema, Delft, The Netherlands.

Shahinpoor, M. (1983). *Advances in the Mechanics and the Flow of Granular Materials*. Gulf Publishing Company, Houston.

Walton, K. (1987). The effective elastic moduli of a random packing of spheres. *J. Mech. Phys. Solids* **35**(3), 213–226.